Humanizing Calculus

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I know how to use the power rule to [find] derivatives, but I don't know where it came from or who made it up or why we have to use it—except that this is what the book and teacher wants us to do.
—Calculus student, spring 2005

The student quoted above raises a concern about the way in which some mathematics topics are still being presented today. As teachers, we sometimes present formulas and rules but do not take the time to talk about the evolution of the mathematics as a human invention. The history of mathematics can supply the why, where, and how for many concepts that are studied (Swetz 1995).

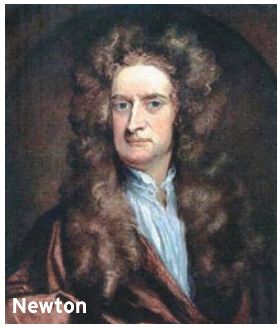
Classroom teachers can help their students develop an appreciation of the invention of calculus in many ways. To address the concerns raised by the student in the opening quotation, I sprinkle quotations and anecdotes throughout a calculus course, some of which I have included in this article. Stories about the invention of calculus by Sir Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716) can add a zesty backdrop to your calculus lessons and thus help students come to see mathematics as a body of knowledge developed by human beings.

WHO INVENTED CALCULUS?

On the first day of class, I give this homework assignment: Find out who invented calculus. A basic Internet search will turn up the names of Newton and Leibniz. Students who investigate further may realize that a long line of mathematicians contributed to the development of calculus. Beginning with the work of Eudoxus in the fourth century BCE and spanning about 2000 years, the groundwork was laid for the invention of calculus. As Hellman (1998) states, neither Newton nor Leibniz created calculus "out of the thin air" (p. 42). Rather, because of the contributions made by other mathematicians, such as Archimedes, Kepler, Descartes, Fermat, Pascal, and Barrow, the basic components of their versions of calculus were already in existence. Any number of mathematicians (if they were alive today) might argue, based on their accomplishments, that they invented calculus. A possible research project for students would be to determine the contributions made by the predecessors of Newton and Leibniz. Students may even stage a debate among themselves as an activity or project in the month that follows the Advanced Placement exam.

CONTEMPORARY GENIUSES

Both Newton and Leibniz are today considered to be among the greatest geniuses ever to have lived. Students are usually interested to learn that they came from vastly different backgrounds. Their fathers died while Newton and Leibniz were very young, but the similarities between them end there. Newton grew up on a farm, and his father was illiterate, signing his name with an X. Leibniz came from an educated family (Bardi 2006). His father was a professor of moral philosophy. Leibniz proudly claimed that he was self-taught, reading from his father's library and attending college at age 14. He earned his bachelor's *and* master's degrees in only three years and completed a doctorate in law by the age of 20. Newton, on the other hand, was less than a shining star as a youngster, and his mother took him out of school at age 16 to work on the family farm. He eventually returned to school and attended Cambridge University. It was during the Great Plague, while the university was closed, that Newton, working in isolation, laid the groundwork for his work in physics, optics, astronomy, and mathematics. Leibniz, on the other hand, whom Frederick the Great of Prussia dubbed "a whole academy unto himself," did not begin his serious study of mathematics until he was on a diplomatic mission in Paris in 1672 (Dunham 1991). It is no surprise that two contemporary geniuses of this magnitude might have had an intellectual spat or two.


Newton


Leibniz

THE PRIORITY DISPUTE

The term *priority dispute* refers to the debate over who invented calculus. Historians of mathematics today generally agree that Newton and Leibniz both invented calculus. Newton is credited as being the first inventor of calculus (1665–1666), and Leibniz is recognized as having independently invented calculus (1675) and as being responsible for its dissemination and for developing the notation that is most similar to that used in modern calculus texts. According to Katz (1993), Newton and Leibniz (not Fermat, Barrow, or their predecessors) are both considered the *inventors* of calculus for four reasons:

1. They each developed general concepts. Newton utilized the fluxion (velocity or rate of change) and fluent (flowing quantity), while Leibniz used the differential and integral. These ideas are related to the two basic problems of calculus: extrema and area.
2. They developed notation and algorithms.
3. They took their respective concepts and applied the inverse relationship.
4. They used these two concepts and applied them to previously unsolvable problems.

The most compelling piece of evidence that clears Leibniz from an accusation of plagiarism actually came from Newton himself. In his first edition of *Principia* (1687), Newton wrote:

> In letters which went between me and that most excellent geometer, G.W. Leibniz, 10 years ago, when I signified I was in the knowledge of a method of determining maxima and minima, of drawing tangents, and the like . . . that most distinguished man wrote back that he had also fallen on a method of the same kind, and communicated his method which hardly differed from mine, except in his form and words and symbols. (Gjertsen 1986, p. 469)

Clearly, Leibniz did not steal his method from Newton, but by the third edition of *Principia* (1726), ten years after Leibniz's death, Newton had removed any reference to Leibniz or his work. Perhaps Leibniz would have avoided these accusations had he acknowledged the correspondence between himself and Newton, as Newton had done in the first edition of the *Principia.*

Although Newton developed his calculus six years before Leibniz even began his serious study of mathematics, Newton failed to publish his work. Newton, however, believed that a scientist's priority derives from having *done* the work and not from the *publication* of the discovery (Hellman 1998). This belief produced much conflict and was the source of consternation for many years to come.

There are various opportunities throughout a calculus course to bring in Leibniz and Newton. In the next section, I will provide examples that can be used when teaching the product and quotient rules and implicit differentiation.

CALCULUS HISTORY FOR SPECIFIC TOPICS

Derivatives in Early Calculus—The Product and Quotient Rules

After students learn the basic formulas for differentiation of sums and differences

$$\left[\frac{d}{dx}(f \pm g) = \frac{d}{dx}f \pm \frac{d}{dx}g\right],$$

they should be asked to think about what a product or quotient rule might look like (Cupillari 2004). It is likely that students will suggest product and quotient rules of the forms:

$$\frac{d}{dx}fg = \left(\frac{d}{dx}f\right)\left(\frac{d}{dx}g\right)$$

and

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}f}{\frac{d}{dx}g}.$$

Leibniz, himself, incorrectly made this assumption before correctly discovering the product and quotient rules (Cupillari 2004). In a manuscript dated November 11, 1675, Leibniz wrote:

> Let us now examine whether $dx\,dy$ is the same thing as $d\,\overline{xy}$, and whether dx/dy is the same thing as
>
> $$d\frac{x}{y};$$
>
> it may be seen that if $y = z^2 + bz$, and $x = cz + d$; then . . . $dy = \overline{2z + b}\beta$. In the same way $dx = +\,c\beta$, and hence $dx\,dy = \overline{2z + bc}\beta^2$. (Child 1920, p. 100)

With an explanation of the notation (we would use dz in place of β and parentheses in place of the bar sign), Leibniz's example is perfectly accessible to calculus students. What is surprising about this example is Leibniz's next statement:

> But you get the same thing if you work out $d\,\overline{xy}$ in a straightforward manner . . . and it is the same thing in the case of divisors. (Child 1920, p. 100)

Leibniz did not work out the details of this claim. Instead, he considered results from the "inverse method of tangents" and discovered his error (Cupillari 2004). The mathematics involved here is not as accessible to beginning calculus students, but through that investigation, Leibniz realized:

> Hence it appears that it is incorrect to say that $dv\,d\psi$ is the same thing as $dv\psi$, or that
>
> $$\frac{dv}{d\psi} = d\frac{v}{\psi};$$
>
> although just above I stated that this was the case, and it appeared to be proved. This is a difficult point. (Child 1920, p. 101)

Rather than going back to his original example, Leibniz provided a counterexample, using $v = x$ and $\psi = x$. Ten days later, in a manuscript dated November 21, 1675, Leibniz provided the correct product and quotient rules. After stating the correct product rule, Leibniz wrote, "Now this is a really noteworthy theorem and a general one for all curves" (Child 1920, p. 107).

Child (1920) points out that, as a logician, Leibniz should have known better than to believe he proved the product rule by providing a single example. A discussion about what counts as "proof" could follow, and students could use the correct product rule to determine that the true value of $d(xy)$ in Leibniz's original problem should have been $(3cz^2 + 2(bc + d)z + bd)dz$. Correcting Leibniz's mistake is a fun way to involve students in the human invention. This example shows students that "calculus was not created in one sequential, correct, and ordered way, as it is presented in textbooks" and that "even a mathematician as brilliant as Leibniz made mistakes when he did not check his work correctly" (Cupillari 2004, p. 195).

When does (fg)′ = f′g′?

While there are an infinite number of counterexamples to the equation $(fg)' = f'g'$, there are also an infinite number of pairs of functions f and g that satisfy the equation. For the trivial cases where f or g is the zero function, or if both f and g are constants, then, of course, $(fg)' = f'g + fg' = f'g'$. As

Maharam and Shaughnessy (1976) pointed out, $(fg)' = f'g'$ is true for any functions f and g, when $f(x) = C(n - x)^{-n}$ and $g(x) = x^n$, where C is an arbitrary nonzero constant and $n \neq 0$. One example of two such functions is $f(x) = 3(2 - x)^{-2}$ and $g(x) = x^2$. Maharam and Shaughnessy (1976) provide a list of other, more interesting pairs of functions for which the "incorrect product rule" produces correct answers. Cupillari (2004) extends this work, showing that an infinite number of function pairs can be found to satisfy the incorrect quotient rule

$$\left(\frac{f}{g}\right)' = \frac{f'}{g'}$$

by letting

$$f(x) = Ce^{\frac{k^2}{k-1}x}$$

with $k \neq 1$ and $g(x) = e^{kx}$. Students can generate pairs of functions and check that they satisfy both the correct and the incorrect product and quotient rules. The fact that these are *recent* publications also demonstrates that mathematics is an evolving, rather than a static, activity.

Differing Approaches and Implicit Differentiation in Early Calculus

Newton's *fluxional calculus* and Leibniz's *differential calculus* had different goals. Leibniz's interests were more philosophical, and his approach was geometric. Newton's interest in calculus was primarily based on motion and the more physical aspects of quantities varying with time. Newton adopted "a new approach in which variables are regarded as flowing quantities generated by the continuous motions of points, lines, etc." (Hollingdale 1989, p. 186). Newton called the variable (e.g., x or y) the *fluent* and its rate of change or velocity the *fluxion,* using a dot (e.g., $\dot{x}$ or $\dot{y}$) to denote the fluxion. Newton realized that the fluxions themselves can be considered fluent quantities, thus having fluxions themselves. These second fluxions of x and y he denoted by $\ddot{x}$ and $\ddot{y}$. As Gjersten (1986) explains in *The Newton Handbook,* to tackle the problems involving the inverse relationship between fluents and fluxions, Newton introduced the notion of a *moment:*

> This was the "indefinitely small" part by which fluents grew in "indefinitely small" periods of time, and was represented by the sign o. The moment of the fluent x would therefore be $\dot{x}o$, and of the fluent y, $\dot{y}o$. In this way, it follows that quantities x and y will become in an indefinitely small interval $x + \dot{x}o$ and $y + \dot{y}o$. (Gjertsen 1986, p. 214)

When learning implicit differentiation, students can implicitly differentiate a problem that was actually posed and solved by Newton: the problem of calculating the tangent to the cubic curve $x^3 - ax^2 + axy - y^3 = 0$. In a tract known as the *Methodus fluxionum et serierum infinitorum* (The Method of Fluxions and Infinite Series), which was written in 1671 but not published until 1736, Newton presented this equation simply as an example, ascribing no significant meaning to this particular curve. However, a curve can be thought of as a path traced by a moving point; and, in this case, the curve Newton presented contains the factors $(x - y)(x^2 + xy - ax + y^2)$, a line and an ellipse. Surely, Newton's interests in cubic curves of this nature stemmed from his interest in science. For example, Newton showed that a planet orbits the sun under an *inverse square law of attraction* moving, not in a circle, but in an ellipse (Hellman 1998). Additionally, his work on the tangent line problem stemmed from his interest in optics and light refraction (Larson, Hostetler, and Edwards 1998).

To differentiate $x^3 - ax^2 + axy - y^3 = 0$, Newton substituted $x + \dot{x}o$ for x and $y + \dot{y}o$ for y to get

$$(x + \dot{x}o)^3 - a(x + \dot{x}o)^2 + a(x + \dot{x}o)(y + \dot{y}o) - (y + \dot{y}o)^3 = 0.$$

The expansion of this equation gives us

$$\begin{aligned} &x^3 + 3x^2\dot{x}o + 3x\dot{x}^2o^2 + \dot{x}^3o^3 - ax^2 - 2ax\dot{x}o - a\dot{x}^2o^2 \\ &\quad + axy + ax\dot{y}o + a\dot{x}yo + a\dot{x}\dot{y}o^2 - y^3 - 3y^2\dot{y}o \\ &\quad - 3y\dot{y}^2o^2 - \dot{y}^3o^3 = 0. \end{aligned}$$

Since $x^3 - ax^2 + axy - y^3 = 0$, then, by substitution, we are left with

$$\begin{aligned} &3x^2\dot{x}o + 3x\dot{x}^2o^2 + \dot{x}^3o^3 - 2ax\dot{x}o - a\dot{x}^2o^2 \\ &\quad + ax\dot{y}o + a\dot{x}yo + a\dot{x}\dot{y}o^2 - 3y^2\dot{y}o \\ &\quad - 3y\dot{y}^2o^2 - \dot{y}^3o^3 = 0. \end{aligned}$$

Since o is an infinitely small quantity, we cast out these terms, leaving

$$3x^2\dot{x} - 2ax\dot{x} + ax\dot{y} + a\dot{x}y - 3y^2\dot{y} = 0.$$

Newton ended here, but the method will be more familiar to us if we continue a bit further. Solving for

$$\frac{\dot{y}}{\dot{x}},$$

we get

$$\frac{\dot{y}}{\dot{x}} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}.$$

Note that if we were to use modern methods to differentiate implicitly, differentiating the third term would require use of the product rule. The product rule is implied with Newton's method, however.

In contrast, Leibniz was interested in finding an appropriate notation to represent thoughts and ways of combining these—a process of computation or reasoning using symbols. As mentioned earlier, Leibniz developed a product rule and a quotient rule. He derived other differentiation rules that look similar to the notation we use today, for example, $d(ax) = a\,dx$ (Eves 1983). In addition, Leibniz used the expressions dx and dy to indicate the difference of two infinitely close values of x and y, respectively, and dy/dx to indicate the ratio of these two values (Cooke 1997). He also used the symbol $\int$ to represent the idea of sum (Katz 1993) as well as the term function (Larson, Hostetler, and Edwards 1998). Had Leibniz used his rules on the problem presented by Newton, his solution would have looked more like our modern implicit differentiation. Students can use modern notation to verify Newton's solution. Because Leibniz's notation was considered superior and because he *published* his work in 1684, it is his notation, rather than Newton's, that is found in today's modern calculus books.

CONCLUSION

Many examples from history demonstrate that calculus was the creation of human beings. I encourage you to explore even further. Other topics that can be visited include the history behind L'Hôpital's Rule or the controversy over lack of rigor in Newton's and Leibniz's calculus. This lack of rigor (later filled in by Cauchy's delta-epsilon process) was considered a major problem by eighteenth-century mathematicians. The contributions and rivalry of the Bernoulli brothers add more fun and humorous stories to the colorful history of calculus. The two brothers were contemporaries of Newton and Leibniz and stout defenders of Leibniz in the priority dispute (Dunham 1991). Multicultural connections can be made to the Islamic and Indian mathematicians in a discussion about sine and cosine power series.

By including facts and anecdotes about the mathematicians who created mathematics, teachers can help calculus and mathematics come alive. Jay Lemke (1990), author of *Talking Science,* claims that students are three to four times as likely to be "highly attentive to 'humanized' science talk as they would be to 'normal' science talk in the classroom" (1990, p. 136). Surely the same can be said for "humanized" calculus.

BIBLIOGRAPHY

Bardi, Jason Socrates. *The Calculus Wars.* New York: Thunder's Mouth Press, 2006.

Child, J. M. *The Early Mathematical Manuscripts of Leibniz.* Chicago: Open Court, 1920.

Cooke, Robert. *The History of Mathematics: A Brief Course.* New York: John Wiley & Sons, 1997.

Cupillari, Antonella. "Another Look at the Rules of Differentiation." *Primus: Problems, Resources, and Issues in Mathematics Undergraduate Studies* 14 (2004): 193–200.

Dunham, William. *Journey through Genius: The Great Theorems of Mathematics.* New York: Penguin Books, 1991.

———. *The Calculus Gallery.* Princeton, NJ: Princeton University Press, 2005.

Eves, Howard. *Great Moments in Mathematics (after 1650)*. Vol. 7. Washington DC: Mathematical Association of America, 1983.

Gjertsen, Derek. *The Newton Handbook.* New York: Routledge & Kegan Paul, 1986.

Hellman, Hal. *Great Feuds in Science: Ten of the Liveliest Disputes Ever.* New York: John Wiley & Sons, 1998.

Hollingdale, Stuart. *Makers of Mathematics.* New York: Penguin Books, 1989.

Katz, Victor J. *A History of Mathematics: An Introduction.* New York: Harper Collins College Publishers, 1993.

Larson, Roland E., Robert P. Hostetler, and Bruce H. Edwards. *Calculus with Analytic Geometry.* New York: Houghton Mifflin, 1998.

Lemke, Jay. *Talking Science.* Westport, CT: Ablex Publishing, 1990.

Maharam, Lewis G., and Edward P. Shaughnessy. "When Does $(fg)' = f'g'$?" *College Mathematics Journal* 7(1976): 38–39.

Newton, Isaac. *The Method of Fluxions and Infinite Series.* 1740. Translated by J. Colson. 2nd ed. Ann Arbor, MI: Xerox University Microforms, 1976.

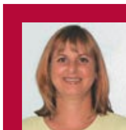
Swetz, Frank. "Some Not So Random Thoughts about the History of Mathematics—Its Teaching, Learning, and Textbooks." *Primus* 5 (June 1995): 97–107.

OTHER RESOURCES

courses.science.fau.edu/~rjordan/phy1931/NEWTON/newton.htm

www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Newton.html

www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Leibniz.html ∞



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